CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Statements About Probability

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Evaluating Statements About Probability

MATHEMATICAL GOALS

This lesson unit addresses common misconceptions relating to probability of simple and compound events. The lesson will help you assess how well students understand concepts of:

- Equally likely events.
- Randomness.
- Sample sizes.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7SP: Investigate chance processes and develop, use, and evaluate probability models

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work, and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction, pairs of students work together to justify or refute mathematical statements. They do this using their own examples and counterexamples. Students then explain their reasoning to another group of students.
- In a whole-class discussion students review the main mathematical concepts of the lesson.
- Students return to their original assessment tasks, and try to improve their own responses.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *Are They Correct*?, a mini-whiteboard, a pen, and an eraser.
- Each pair of students will need a copy of the sheet *True, False, or Unsure*?, (cut up into cards), a large sheet of paper for making a poster, and a glue stick.
- There is a projector resource to support a whole-class discussion.

TIME NEEDED

15 minutes before the lesson and a 1-hour lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task*: Are They Correct?* (15 minutes)

Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work. You will then be able to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give out the assessment task *Are They Correct?*

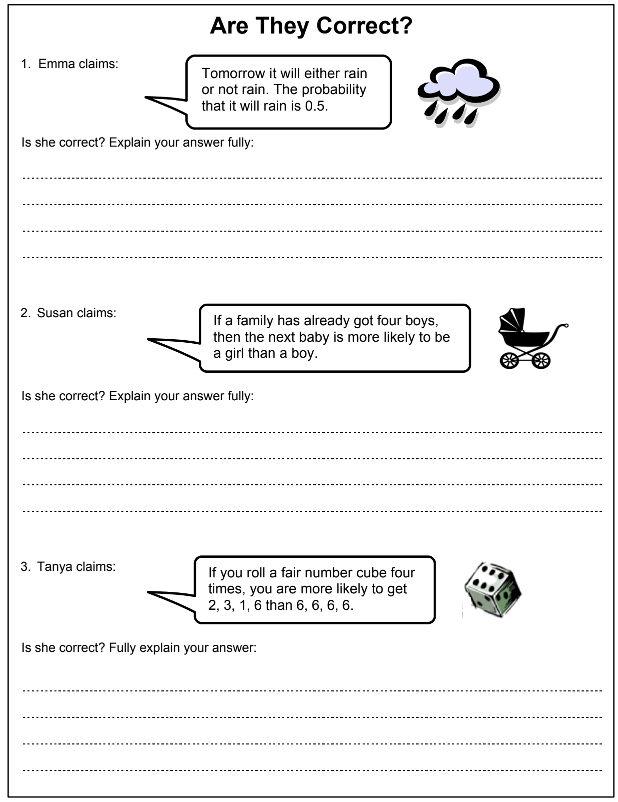
Briefly introduce the task and help the class to understand each problem.

Read through each statement and make sure you understand it.

Try to answer each question as carefully as you can.

Show all your work so that I can understand your reasoning.

It is important that, as far as possible, students are allowed to answer the questions without your assistance.



Are They Correct?

1. Emma claims:

Tomorrow it will either rain or not rain. The probability that it will rain is 0.5.

Is she correct? Explain your answer fully:

2. Susan claims:

If a family has already got four boys, then the next baby is more likely to be a girl than a boy.

Is she correct? Explain your answer fully:

3. Tanya claims:

If you roll a fair number cube four times, you are more likely to get 2, 3, 1, 6 than 6, 6, 6, 6.

Is she correct? Fully explain your answer:

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.

Common issues:	Suggested questions and prompts:
Q1. Student assumes that both outcomes are equally likely	• What factors affect whether it will rain tomorrow? • What does a probability of 0.5 mean?
Q2. Student assumes that later random events 'compensate' for earlier ones For example: The student argues that if there are already four boys in the family, the next will is likely to be a girl.	• What is the probability that the baby will be a girl? • Does the fact that there are already four boys in the family affect the sex of the next child?
Q3. Student relies on their own experience For example: The student states they have never thrown four sixes in a row.	• Is it more difficult to throw a six than a two? • Is it more difficult to throw a six, then another six or a two, then a three?

SUGGESTED LESSON OUTLINE

Whole-class interactive introduction (10 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.

Display slide P-1 of the projector resources.

Ensure the students understand the problem:

> *Your task is to decide if the statement is true.*
>
> *Once you have made a decision you need to convince me.*

Two bags of jellybeans

I have two bags. Both contain red and yellow jellybeans.

There are more red jellybeans in bag **A** than in bag **B**.

If I choose one jellybean from each bag I am more likely to choose a red one from bag **A** than from bag **B**.

Allow students a few minutes to think about the problem individually, then a further few minutes to discuss their initial ideas in pairs. Ask students to write their explanations on their whiteboard.

If students are unsure, encourage them to think of a simple experiment that could simulate the statement:

> *Do you know how many red and yellow jellybeans are in each bag? Give me an example of the numbers of jellybeans in each bag. Draw a picture of the situation.*
>
> *Can you think of a situation for which the statement is true? [For example, two red jellybeans and one yellow jellybean in bag A and one red jellybean and one yellow jellybean in bag B.]*
>
> *Can you think of a situation for which the statement is false? [For example, two red jellybeans and three yellow jellybeans in bag A and one red jellybean and one yellow jellybean in bag B.]*

Ask students to show you their mini-whiteboards. Select two or three students with different answers to explain their reasoning on the board. Encourage the rest of the class to comment.

Then ask:

> *Chen, can you rewrite the statement so that it is always true?*
>
> *Carlos, do you agree with Chen's explanation? Put Chen's explanation into your own words.*
>
> *Does anyone have a different statement that is also always true?*

This statement highlights the misconception that students often think the results of random selection are dependent on numbers rather than ratios.

Collaborative activity (20 minutes)

Organize the class into pairs of students.

Give each pair a copy of *True, False, or Unsure*?, a large piece of paper for making a poster, and a glue stick.

Ask students to divide their paper into two columns: one for statements they think are true, and the other for statements they think are false.

Ask students to take each statement in turn:

Select a card and decide whether it is a true or false statement.

Convince your partner of your decision.

It is important that you both understand the reasons for the decision. If you don't agree with your partner, explain why. You are both responsible for each other's learning.

If you are both happy with the decision, glue the card onto the paper. Next to the card, write reasons to support your decision.

Put to one side any cards you are unsure about.

You may want to use slide P-2 of the projector resource to display these instructions

You have two tasks during small group work: to make a note of student approaches to the task, and to support student problem solving.

Make a note of student approaches to the task

Notice how students make a start on the task, whether they get stuck, and how they respond if they do come to a halt. For example, are students drawing diagrams, working out probabilities, or simply writing a description? As they work on the task, listen to their reasoning carefully and note misconceptions that arise for later discussion with the whole class.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, write a relevant question on the board and hold a brief whole-class discussion.

Here are some questions you may want to ask your students:

Card A: This statement addresses the misconception that probabilities give the proportion of outcomes that *will* occur.

Is it possible to get five sixes in a row with a fair six-sided number cube?

Is it more difficult to roll a six than, say, a two?

Card B: This statement addresses the misconception that 'special' events are less likely than 'more representative' events. Students often assume that selecting an 'unusual' letter, such as W, X, Y or Z is a less likely outcome.

Is the letter X more difficult to select than the letter T?

Are the letters W and X more difficult to select than the letters D and T?

Card C: This statement addresses the misconception that later random events 'compensate' for earlier ones.

Does the coin have a memory?

Card D: This statement addresses the misconception that all outcomes are equally likely, without considering that some are much more likely than others.

Is the probability of a local school soccer team beating the World Cup champions $\frac{1}{3}$*?*

Card E This statement addresses the misconception that all outcomes are equally likely, without considering that some are much more likely than others. Students often simply count the different outcomes.

Are all three outcomes equally likely? How do you know? How can you check your answer?

What are all the possible outcomes when two coins are tossed? How does this help?

Card F: This statement addresses the misconception that the two outcomes are equally likely.

How can you check your answer?

In how many ways can you score a three? In how many ways can you score a two?

Card G: This statement addresses the misconception that probabilities give the proportion of outcomes that *will* occur.

When something is certain, what is its probability?

What experiment could you do to check if this answer is correct? [One student writes the ten answers e.g. false, true, true, false, true, false, false, false, true, false. Without seeing these answers the other student guesses the answers].

Card H: This statement addresses the misconception that the sample size is irrelevant. Students often assume that because the probability of one head in two coin tosses is $\frac{1}{2}$, the probability of n heads in $2n$ coin tosses is also $\frac{1}{2}$.

Is the probability of getting one head in two coin tosses $\frac{1}{2}$*? How do you know?*

Show me a possible outcome if there are four coin tosses. Show me another. How many possible outcomes are there? How many outcomes are there with two heads?

Sharing work (5 minutes)

As students finish the task, ask them to compare work with a neighboring pair.

Check which answers are different.

A member of each group needs to explain their reasoning for these answers. If anything is unclear, ask for clarification.

Then together consider if you should change any of your answers.

It is important that everyone in both groups understands the math. You are responsible for each other's learning.

Whole-class discussion (10 minutes)

Organize a discussion about what has been learned. Focus on getting students to understand the reasoning, not just checking that everyone produced the same answers.

Ask students to choose one card they are certain is true and to explain why they are certain to the rest of the class. Repeat this with the statements that students believe are false. Finally, as a whole class, tackle the statements that students are not so sure about.

Ben, why did you decide this statement was true/false?

Does anyone agree/disagree with Ben?

Does anyone have a different explanation to Ben's?

In addition to asking for a variety of methods, pursue the theme of listening and comprehending each others' methods by asking students to rephrase each other's reasoning.

Danielle, can you put that into your own words?

You may also want to ask students:

Select two cards that use similar math. Why are they similar? Is there anything different about them? [Students are likely to select cards D and E.]

In trials, students have found card H challenging.

What are the possible outcomes? Have you listed all of the outcomes? Have you listed all the outcomes where there are two heads? What does this show?

Improving individual solutions to the assessment task (10 minutes)

Return the original assessment task *Are they Correct?* to students, together with a blank copy of the task.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

If you find you are running out of time, then you could set this task in the next lesson, or for homework.

SOLUTIONS

Assessment Task: *Are They Correct?*

1. This statement is incorrect. It highlights the misconception that all events are equally likely. There are many factors (e.g., the season) that will influence the chances of it raining tomorrow.
2. *Assuming that the sex of a baby is a random, independent event equivalent to tossing a coin*, the statement is incorrect. It highlights the misconception that later random events can 'compensate' for earlier ones. The assumption is important: there are many beliefs and anecdotes about what determines the sex of a baby, but 'tossing a coin' turns out to be a reasonably good model[1].
3. This statement is incorrect. This highlights the misconception that 'special' events are less likely than 'more representative' events.

Collaborative Activity: *True, False, or Unsure*

A. If you roll a six-sided number cube, and it lands on a six more than any other number, then the number cube must be biased.

False. This statement addresses the misconception that probabilities give the proportion of outcomes that **will** occur. With more information (**How many** times was the cube rolled? **How many** more sixes were thrown?) more advanced mathematics could be used to calculate the **probability** that the dice was biased, but you could never be 100% certain.

B. When randomly selecting four letters from the alphabet, you are more likely to come up with

D, T, M, J than W, X, Y, Z.

False. This highlights the misconception that 'special' events are less likely than 'more representative' events. Students often assume that selecting the 'unusual' letters W, X, Y and X is less likely.

C. If you toss a fair coin five times and get five heads in a row, the next time you toss the coin it is more likely to show a tail than a head.

False. This highlights the misconception that later random events 'compensate' for earlier ones. The statement implies that the coin has some sort of 'memory'. People often use the phrase 'the law of averages' in this way.

D. There are three outcomes in a soccer match: win, lose, or draw. The probability of winning is therefore $\frac{1}{3}$.

False. This highlights the misconception that all outcomes are equally likely, without considering that some are much more likely than others. The probabilities are dependent on the rules of the game and which teams are playing.

[1] See, for example, http://www.bbc.co.uk/news/magazine-12140065

E. When two coins are tossed there are three possible outcomes: two heads, one head, or no heads. The probability of two heads is therefore $\frac{1}{3}$.

False. This highlights the misconception that all outcomes are equally likely, without considering that some are much more likely than others. There are four equally likely outcomes: HH, HT, TH, TT. The probability of two heads is $\frac{1}{4}$.

F. Scoring a total of three with two number cubes is twice as likely as scoring a total of two.

True. This highlights the misconception that the two outcomes are equally likely. To score three there are two outcomes, 1,2 and 2,1, but to score two there is only one outcome, 1,1.

G. In a 'true or false?' quiz with ten questions, you are certain to get five correct if you just guess.

False. This highlights the misconception that probabilities give the exact proportion of outcomes that will occur. If a lot of people took the quiz, you would expect the mean score to be *about* 5, but the individual scores would vary.

Probabilities do not say for certain what will happen, they only give an indication of the likelihood of something happening. The only time we can be certain of something is when the probability is 0 or 1.

H. The probability of getting exactly two heads in four coin tosses is $\frac{1}{2}$.

False. This highlights the misconception that the sample size is irrelevant. Students often assume that because the probability of one head in two coin tosses is $\frac{1}{2}$, then the probability of n heads in $2n$ coin tosses is also $\frac{1}{2}$. In fact the probability of two out of four coin tosses being heads is $\frac{6}{16}$.

This can be worked out by writing out all the sixteen possible outcomes:

HHHH, HHHT, HHTH, HTHH, THHH, TTTT, TTTH, TTHT, THTT, HTTT, HHTT, HTTH, TTHH, THTH, HTHT, THHT.

This may be calculated from Pascal's triangle:

	1		
2 coins	1 **2** 1	4 outcomes	Probability(1 head) =
	1 3 3 1		
4 coins	1 4 **6** 4 1	16 outcomes	Probability(2 heads)
	1 5 10 10 5 1		
6 coins	1 6 15 **20** 15 6 1	64 outcomes	Probability(3 heads)

Students are not expected to make this connection!

Are They Correct?

1. Emma claims:

> Tomorrow it will either rain or not rain. The probability that it will rain is 0.5.

Is she correct? Explain your answer fully:

2. Susan claims:

> If a family has already got four boys, then the next baby is more likely to be a girl than a boy.

Is she correct? Explain your answer fully:

3. Tanya claims:

> If you roll a fair number cube four times, you are more likely to get 2, 3, 1, 6 than 6, 6, 6, 6.

Is she correct? Fully explain your answer:

Card Set: True, False Or Unsure?

A.
If you roll a six-sided number cube, and it lands on a six more than any other number, then the number cube must be biased.

B.
When randomly selecting four letters from the alphabet, you are more likely to come up with

D, T, M, J

than

W, X, Y, Z.

C.
If you toss a fair coin five times and get five heads in a row, the next time you toss the coin it is more likely to show a tail than a head.

D.
There are three outcomes in a soccer match: win, lose, or draw. The probability of winning is therefore $\frac{1}{3}$.

E.
When two coins are tossed there are three possible outcomes: two heads, one head, or no heads. The probability of two heads is therefore $\frac{1}{3}$.

F.
Scoring a total of three with two number cubes is twice as likely as scoring a total of two.

G.
In a “true or false?” quiz with ten questions, you are certain to get five correct if you just guess.

H.
The probability of getting exactly two heads in four coin tosses is $\frac{1}{2}$.

Two bags of jellybeans

I have two bags. Both contain red and yellow jellybeans.

There are more red jellybeans in bag **A** than in bag **B**.

If I choose one jellybean from each bag I am more likely to choose a red one from bag **A** than from bag **B**.

True, False or Unsure?

- Take turns to select a card and decide whether it is a true or false statement.
- Convince your partner of your decision.
- It is important that you both understand the reasons for the decision. If you don't agree with your partner, explain why. You are both responsible for each other's learning.
- If you are both happy with the decision, glue the card onto the paper. Next to the card, write reasons to support your decision.
- Put to one side any cards you are unsure about.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee